The effect of non-linearities on statistical distributions in the theory of sea waves

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(Received 15 March 1963)

The statistical density function is derived for a variable (such as the surface elevation in a random sea) that is 'weakly non-linear'. In the first approximation the distribution is Gaussian, as is well known. In higher approximations it is shown that the distribution is given by successive sums of a Gram–Charlier series; not quite in the form that has sometimes been used as an empirical fit for observed distributions, but in a modified form due to Edgeworth.

It is shown that the cumulants of the distribution are much simpler to calculate than the corresponding moments; and the approximate distributions are in fact derived by inversion of the cumulant-generating function.

The theory is applied to random surface waves on water. The third cumulant and hence the skewness of the distribution of surface elevation is evaluated explicitly in terms of the directional energy spectrum. It is shown that the skewness λ_3 is generally positive, and positive upper and lower bounds for λ_3 are derived. The theoretical results are compared with some measurements made by Kinsman (1960).

It is found that for free, undamped surface waves the skewness of the distribution of surface *slopes* is of a higher order than the skewness of the surface elevation. Hence the observed skewness of the slopes may be a sensitive indicator of energy transfer and dissipation within the water.

1. Introduction

It is well known that in the linear theory of wind-generated water waves, in which squares and higher powers of the surface displacement are neglected, the statistical distribution of the surface elevation and its derivatives is Gaussian, under quite general conditions. Moreover, the Gaussian distribution of the surface elevation and bottom pressure has been fairly well verified in some circumstances (see, for example, Rudnick 1950; Barber 1950; Pierson 1955; MacKay 1959). Quite early, however, Birkhoff & Kotik (1952) pointed out significant departures from the Gaussian distribution for waves in shallow water. Similar, though less pronounced, effects for waves in deep water were found by Burling (1955) and Kinsman (1960).

The distribution of surface slopes was shown by Cox & Munk (1956) to have an appreciable skewness in the direction of the wind; and surface curvatures in wind-generated waves may be even more radically non-Gaussian (Schooley 1955).

Several theoretical investigations have lately been made into the dynamical effects which non-linearities produce in the quadratic spectrum of the sea surface (Tick 1959, 1961; Phillips 1960, 1961; Hasselmann 1960, 1961, 1962). However, the effect of such non-linearities on the statistical distributions has received less attention. Phillips (1961) has pointed out that the surface elevation must in fact have a coefficient of skewness of the same order of magnitude as the surface slope; but the higher moments of the distribution have not been calculated, nor has the complete distribution been derived.

Some authors (Cox & Munk 1956; Kinsman 1960) have fitted the observed distributions of surface slope or elevation by means of a Gram–Charlier series, with apparently no justification beyond the fact that any function that is sufficiently well-behaved can be expanded in such a series. The coefficients of successive terms are related to the moments of the function itself.

In the present paper we derive the theoretical distribution of quantities (such as the surface elevation or surface slope) which can be described as 'weakly non-linear', that is to say the ordinary representation as the sum of independent random components is valid to a first approximation, but quadratic and higher-order interactions between the components cannot be entirely neglected. At each stage the calculation is carried uniformly to a certain power of the component amplitudes.

As one would expect, the first approximation corresponds to the ordinary Gaussian distribution. It is found that higher approximations are described by the Gaussian law multiplied by certain polynomials. These expressions correspond in fact to successive terms in a Gram–Charlier series; not, however, in the form that has been commonly used for fitting the distributions, but in a modified form due to Edgeworth (1906*a*, *b*, *c*). For example, in the second approximation a cubic polynomial occurs, but in the third approximation one must include a quartic polynomial plus another of degree six.

Roughly the method is as follows: from the dynamical equations it is possible to calculate successively higher *moments* of the statistical variable. It turns out that certain combinations of the moments, namely, the *cumulants*, are simpler to calculate, and just as convenient to handle, as the moments themselves. By calculating the cumulant-generating function to a certain order and taking the Fourier transform one obtains the desired approximation to the distribution function.

The analysis is essentially similar to Edgeworth's (1906) generalization of the Gaussian 'law of error' for a single variable, but is presented here in a rather different form and is moreover extended to two or more dependent variables.

The device of truncating the cumulant-generating function has been used in the analytical theory of turbulence (see, for example, O'Brien & Francis 1962) but not, so far as the author is aware, for the specific purpose of calculating probability densities. Hence some of the results of the present study may be applicable also to turbulent fluctuations.

The basic analysis for a single non-linear variable is given in § 2. This is then applied, in § 3, to the distribution of surface elevation in a random sea. In § 4 the results are compared with recent observations made by Kinsman (1960).

The next three sections follow a similar scheme: the joint distribution of two related non-linear variables is derived in § 5; this is applied to the joint distribution of surface slopes in § 6, and in § 7 the well-known observations of Cox & Munk (1956) are discussed. The conclusions are restated in § 8.

2. A single non-linear variable

As will be shown in § 3, the linear spectral representation of the sea surface elevation ζ can be expressed in the form

$$\zeta = \sum_{i=1}^{N} \alpha_i \xi_i,$$

where the α_i are constants and the ξ_i are independent random variables symmetrically distributed about 0 with variance V_i, say. The convergence of $p(\zeta)$ to a Gaussian distribution, with variance ΣV_i, is a case of the so-called 'law of large numbers'.

In the more exact non-linear theory, in order to satisfy the dynamical equations for ζ, quadratic and higher-order terms must be added to ζ. Let us then consider, in a general way, the distribution of the variable

$$\zeta = \alpha_i \xi_i + \alpha_{ij} \xi_i \xi_j + \alpha_{ijk} \xi_i \xi_j \xi_k + \ldots, \tag{2.1}$$

where α_i, α_{ij}, α_{ijk}, etc., are constants, and the summation convention is used. Thus in (2.1) each product is summed over all repeated suffices, from 1 to N. With each value of i is associated a vector $\mathbf{u}_i$ (the wavenumber). Later, we shall make $N \to \infty$ and each $V_i \to 0$ in such a way that over any small but fixed region $d\mathbf{u}$

$$\sum_{\mathbf{u} \ni d\mathbf{u}} V_i \to F(\mathbf{u})\, d\mathbf{u} + O(d\mathbf{u})^2. \tag{2.2}$$

The first few moments of ζ can be written down by inspection. Thus taking mean values in (2.1) one has

$$\bar{\zeta} = \alpha_i \bar{\xi_i} + \alpha_{ij} \overline{\xi_i \xi_j} + \alpha_{ijk} \overline{\xi_i \xi_j \xi_k} + \ldots.$$

All mean values of odd-order terms vanish, while among the terms of even order only those remain in which each ξ_i is paired with a similar ξ_i. Thus*

$$\bar{\zeta} = \alpha_{ii} V_i + 3\alpha_{iijj} V_i V_j + \ldots. \tag{2.3}$$

(It is assumed that the α are symmetric in their suffices so that, for example, $\alpha_{ijij} = \alpha_{ijji} = \alpha_{iijj}$.) There are, in general, terms involving $\overline{\xi_i^4}$, $\overline{\xi_i^6}$, etc.; these become negligible on passing to the limit as $N \to \infty$, and so will be ignored.

In a similar way, by squaring both sides of (2.1) one has

$$\zeta^2 = (\alpha_i \xi_i + \alpha_{ij} \xi_i \xi_j + \ldots)(\alpha_k \xi_k + \alpha_{kl} \xi_k \xi_l + \ldots)$$

and on taking mean values

$$\overline{\zeta^2} = \alpha_i \alpha_i V_i + (2\alpha_{ij}\alpha_{ji} + \alpha_{ii}\alpha_{jj}) V_i V_j + 6\alpha_i \alpha_{ijj} V_i V_j + \ldots. \tag{2.4}$$

The higher moments may be calculated similarly, but a direct approach leads to complications. We shall show how these can be circumvented.

* The usual summation convention is extended to three repeated indices.

It will be noticed that some of the terms in (2.4) and in higher moments can be factorized*

$$\alpha_{ii}\alpha_{jj}V_iV_j = (\alpha_{ii}V_i)(\alpha_{jj}V_j),$$

but other terms, e.g. $\alpha_{ij}\alpha_{ji}V_iV_j$ cannot. The latter terms may be called 'irreducible', and it will be convenient to introduce an abbreviated notation for them. Thus let $\alpha_{ij\ldots l}$ be denoted shortly by A_r, where r is the number of suffices $i, j \ldots l$; and let the sum of all the irreducible terms in the mean product

$$\overline{(A_p\xi_{i_1}\ldots\xi_{i_p})(A_q\xi_{j_1}\ldots\xi_{j_q})\ldots(A_s\xi_{l_1}\ldots\xi_{l_s})}$$

be denoted simply by

$$(A_pA_q\ldots A_s).$$

Clearly when $(p+q+\ldots+s)$ is odd, the above expression vanishes. Also

$$(A_1^2) = \alpha_i\alpha_iV_i, \qquad (A_2) = \alpha_{ii}V_i, \tag{2.5}$$

$$\left.\begin{aligned}&(A_1^n) = 0 \quad (n \geqslant 4),\\ &(A_1^2A_2) = 2\alpha_i\alpha_j\alpha_{ij}V_iV_j, \qquad (A_1A_3) = 3\alpha_i\alpha_{ijj}V_iV_j,\\ &(A_2^2) = 2\alpha_{ij}\alpha_{ji}V_iV_j, \qquad (A_4) = 3\alpha_{iijj}V_iV_j,\end{aligned}\right\} \tag{2.6}$$

$$\left.\begin{aligned}&(A_1^2A_2^2) = 8\alpha_i\alpha_j\alpha_{ik}\alpha_{jk}V_iV_jV_k, \qquad (A_1^3A_3) = 6\alpha_i\alpha_j\alpha_k\alpha_{ijk}V_iV_jV_k,\\ &(A_1^2A_4) = 12\alpha_i\alpha_j\alpha_{ijkk}V_iV_jV_k, \qquad (A_1A_5) = 15\alpha_{iijjkk}V_iV_jV_k,\\ &(A_1A_2A_3) = (6\alpha_i\alpha_{ij}\alpha_{jkk} + 6\alpha_i\alpha_{jk}\alpha_{ijk})V_iV_jV_k, \quad (A_2^3) = 8\alpha_{ij}\alpha_{jk}\alpha_{ki}V_iV_jV_k,\\ &(A_3^2) = (9\alpha_{ijj}\alpha_{ikk} + 6\alpha_{ijk}\alpha_{ijk})V_iV_jV_k, \quad (A_2A_4) = 12\alpha_{ij}\alpha_{ijkk}V_iV_jV_k,\\ &(A_6) = 15\alpha_{iijjkk}V_iV_jV_k.\end{aligned}\right\} \tag{2.7}$$

The first few moments can now be written shortly as

$$\left.\begin{aligned}&\bar{\zeta} = \sum_p (A_p), \qquad \overline{\zeta^2} = \sum_{p,q}[(A_pA_q) + (A_p)(A_q)],\\ &\overline{\zeta^3} = \sum_{p,q,r}[(A_pA_qA_r) + 3(A_pA_q)(A_r) + (A_p)(A_q)(A_r)],\end{aligned}\right\} \tag{2.8}$$

etc., where the summations are over all positive integral values of p, q, r (including equal values). Generally,

$$\overline{\zeta^n} = \sum_{p,q,\ldots,s}[C(n)\varpi(n) + C(n-1,1)\varpi(n-1,1) + \ldots], \tag{2.9}$$

where $\varpi(i,j,\ldots,l)$ denotes some grouping of A_p, A_q, …, A_s into unordered† sets containing i, j, …, l members, and $C(i,j,\ldots,l)$ denotes the number of ways of choosing such sets. If r denotes the number of sets in ϖ we have

$$C(i,j,\ldots,l) = \frac{1}{r!}\,\frac{n!}{i!\,j!\ldots l!}. \tag{2.10}$$

* The factorization of any given product can be shown to be unique.

† Both the sets and the members of each set are unordered. Each ϖ is considered as distinct from the rest.

We have thus calculated $\overline{\zeta^n} = \mu_n$, the nth moment of the distribution. It turns out, however, that the *cumulants* of the distribution are much simpler. Whereas the moments correspond to the coefficients of $(it)^n$ in the function

$$\begin{aligned}\phi(it) &= \int_{-\infty}^{\infty} p(\zeta)\, e^{it\zeta}\, d\zeta \\ &= 1+\frac{\mu_1}{1!}(it)+\frac{\mu_2}{2!}(it)^2+\dots,\end{aligned} \tag{2.11}$$

the cumulants, by definition, correspond to the coefficients of $(it)^n$ in

$$\begin{aligned}\mathrm{K}(it) &= \log\phi(it) \\ &= \frac{\kappa_1}{1!}(it)+\frac{\kappa_2}{2!}(it)^2+\dots.\end{aligned} \tag{2.12}$$

On equating coefficients of $(it)^n$ in (2.12) one has

$$\kappa_1 = \mu_1, \quad \kappa_2 = \mu_2-\mu_1^2, \quad \kappa_3 = \mu_3-3\mu_1\mu_2+2\mu_1^3,$$

etc., and so on substitution from (2.8) we have

$$\kappa_1 = \sum_p (A_p), \quad \kappa_2 = \sum_{p,q} (A_p A_q), \quad \kappa_3 = \sum_{p,q,r} (A_p A_q A_r),$$

etc. This suggests the relation

$$\kappa_n = \sum_{p,q,\dots,s} (A_p A_q \dots A_s). \tag{2.13}$$

To prove this, we note that since

$$\phi(it) = e^{\mathrm{K}(it)} = \exp\left[1+\frac{\kappa_1}{1!}(it)+\frac{\kappa_2}{2!}(it)^2+\dots\right] \tag{2.14}$$

one has, on equating coefficients of $(it)^n$ in this expression,

$$\begin{aligned}\mu_n &= \sum_{r=1}^{\infty}\frac{n!}{r!}\sum_{i+j+\dots+l=n}\frac{\kappa_i\kappa_j\dots\kappa_l}{i!\,j!\dots l!} \\ &= \sum_{r=1}^{\infty}\sum_{i+j+\dots+l=n} C(i,j,\dots,l)\,\kappa_i\kappa_j\dots\kappa_l,\end{aligned} \tag{2.15}$$

where $C(i,j,\dots,l)$ is given by (2.10). Therefore the equations for μ_n in terms of the κ_n are formally identical with the equations for μ_n in terms of $\Sigma(A_p A_q \dots A_s)$. Since $\mu_1 = \Sigma(A_p)$ the general result (2.13) follows by induction.

Retaining all terms up to the sixth order in the ξ_i (i.e. third order in V_i) we have from (2.13)

$$\left.\begin{aligned}\kappa_1 &= (A_2)+(A_4)+(A_6), \\ \kappa_2 &= (A_1^2)+(A_2^2)+(A_3^2)+2(A_1A_3)+2(A_1A_5)+2(A_2A_4), \\ \kappa_3 &= (A_2^3)+3(A_1^2A_2)+3(A_1^2A_4)+6(A_1A_2A_3), \\ \kappa_4 &= (A_1^4)+4(A_1^3A_3)+6(A_1^2A_2^2).\end{aligned}\right\} \tag{2.16}$$

From (2.16) it is seen that κ_1 and κ_2 are both of order V, in general. However, when $n \geqslant 2$ the lowest non-vanishing term in κ_n is $n(A_1^{n-1}A_{n-1})$, which is $O(V^{n-1})$.

The coefficients of skewness and of kurtosis will be defined by

$$\lambda_3 = \kappa_3/\kappa_2^{\frac{3}{2}}, \quad \lambda_4 = \kappa_4/\kappa_2^2, \tag{2.17}$$

which are of order $V^{\frac{1}{2}}$ and V, respectively. More generally,

$$\lambda_n = \kappa_n/\kappa_2^{\frac{1}{2}n} = O(V^{\frac{1}{2}n-1}). \tag{2.18}$$

The density of ζ

Now provided that the probability density $p(\zeta)$ is uniquely determined by its moments, $p(\zeta)$ can be obtained directly from (2.8) by inverting the Fourier transform:

$$p(\zeta) = \frac{1}{2\pi}\int_{-\infty}^{\infty} \phi(it)\, e^{-it\zeta}\, dt = \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp\left[\mathrm{K}(it) - it\zeta\right] dt$$
$$= \frac{1}{2\pi}\int_{-\infty}^{\infty} \exp\left[(\kappa_1 - \zeta)\, it + \tfrac{1}{2}\kappa_2 (it)^2 + \tfrac{1}{6}\kappa_3 (it)^3 + \dots\right] dt.$$

Substituting $\qquad t = s/\kappa_2^{\frac{1}{2}}, \quad (\zeta - \kappa_1) = f\kappa_2^{\frac{1}{2}},$

we have

$$p(\zeta) = \frac{1}{2\pi\kappa_2^{\frac{1}{2}}}\int_{-\infty}^{\infty} \exp\left[-\tfrac{1}{2}(s^2 + 2ifs) + \tfrac{1}{6}\lambda_3 (is)^3 + \tfrac{1}{24}\lambda_4 (is)^4 + \dots\right] ds,$$

where λ_n, as we have seen, is $O(V^{\frac{1}{2}n-1})$. The second group of terms under the exponential can now be expanded in powers of $V^{\frac{1}{2}}$, giving

$$p(\zeta) = \frac{1}{2\pi\kappa_2^{\frac{1}{2}}}\int_{-\infty}^{\infty} \exp\left[-\tfrac{1}{2}(s^2 + 2ifs)\right]\left[1 + \tfrac{1}{6}\lambda_3 (is)^3 + \{\tfrac{1}{24}\lambda_4 (is)^4 + \tfrac{1}{72}\lambda_3^2 (is)^6\} + \dots\right] ds.$$

But we have identically

$$\frac{1}{(2\pi)^{\frac{1}{2}}}\int_{-\infty}^{\infty} \exp\left[-\tfrac{1}{2}(s^2 + 2ifs)\right](is)^n\, ds = \frac{(-1)^n}{(2\pi)^{\frac{1}{2}}}\frac{d^n}{df^n}\int_{-\infty}^{\infty} \exp\left[-\tfrac{1}{2}(s^2 + 2ifs)\right] ds$$
$$= (-1)^n \frac{d^n}{df^n} e^{-\frac{1}{2}f^2} = e^{-\frac{1}{2}f^2} H_n(f),$$

where H_n denotes the Hermite polynomial of degree n:

$$H_n = f^n - \frac{n(n-1)}{1!}\frac{f^{n-2}}{2} + \frac{n(n-1)(n-2)(n-3)}{2!}\frac{f^{n-4}}{2^2} - \dots. \tag{2.19}$$

Hence we have

$$p(\zeta) = (2\pi\kappa_2)^{-\frac{1}{2}} e^{-\frac{1}{2}f^2}\left[1 + \tfrac{1}{6}\lambda_3 H_3 + (\tfrac{1}{24}\lambda_4 H_4 + \tfrac{1}{72}\lambda_3^2 H_6) + \dots\right]. \tag{2.20}$$

From (2.19)

$$\left.\begin{aligned} H_3 &= f^3 - 3f, & H_4 &= f^4 - 6f^2 + 3, \\ H_5 &= f^5 - 10f^3 + 15f, & H_6 &= f^6 - 15f^4 + 45f^2 - 15. \end{aligned}\right\} \tag{2.21}$$

Equation (2.20) is the distribution sought. It corresponds quite closely to Edgeworth's form of the type A Gram–Charlier series (Kendall & Stuart 1958, § 6.18).

In a first approximation, λ_3 and λ_4 can be neglected and we take

$$\kappa_2 = \langle A_1^2\rangle = \alpha_i \alpha_i V_i.$$

Thus $$p(\zeta) \doteqdot e^{-\frac{1}{2}f^2}/(2\pi\kappa_2)^{\frac{1}{2}}, \quad f = \zeta/\kappa_2^{\frac{1}{2}}. \tag{2.22}$$

This is the well-known Gaussian law.

In the next approximation λ_3 is taken into account, but λ_3^2 and λ_4 are neglected. To the same approximation

$$\left.\begin{aligned} \kappa_1 &= (A_2) = \alpha_{ii}V_i, & \kappa_2 &= (A_1^2) = \alpha_i\alpha_i V_i, \\ \kappa_3 &= 3(A_1^2 A_2) = 6\alpha_i\alpha_j\alpha_k V_i V_j, & \kappa_n &= 0, \ (n \geqslant 4), \end{aligned}\right\} \tag{2.23}$$

and so $$p(\zeta) \doteqdot (2\pi\kappa_2)^{-\frac{1}{2}} e^{-\frac{1}{2}f^2}[1 + \tfrac{1}{6}\lambda_3(f^3 - 3f)], \tag{2.24}$$

where $$f = \zeta/\kappa_2^{\frac{1}{2}} - \kappa_1/\kappa_2^{\frac{1}{2}} \tag{2.25}$$

and $$\lambda_3 = 6\alpha_i\alpha_j\alpha_{ij}V_iV_j/(\alpha_i\alpha_j V_i)^{\frac{3}{2}}. \tag{2.26}$$

Thus the mean value of ζ is shifted by an amount $\alpha_{ii}V_i$ and the density is multiplied by the factor

$$[1 + \tfrac{1}{6}\lambda_3(f^3 - 3f)] \tag{2.27}$$

which introduces a skewness λ_3. The kurtosis is zero, as are all the higher cumulants.

In the next approximation, the distribution is given by the full equation (2.20). The mean κ_1, variance κ_2 and skewness λ_3 are all slightly modified, and a non-zero kurtosis λ_4 appears, given by κ_4/κ_2^2, where

$$\begin{aligned} \kappa_4 &\doteqdot 4(A_1^3 A_3) + 6(A_1^2 A_2^2) \\ &= 24\alpha_i\alpha_j\alpha_k\alpha_{ijk}V_iV_jV_k + 48\alpha_i\alpha_j\alpha_{ikjk}V_iV_jV_k. \end{aligned} \tag{2.28}$$

3. Application to gravity waves

Consider a random, homogeneous surface displacement on water of infinite depth. To a first approximation such a surface may be represented in the form $z = \zeta^{(1)}$ where

$$\zeta^{(1)} = \sum_{n=1}^{N'} a_n \cos\psi_n, \quad \psi_n = (\mathbf{k}_n . \mathbf{x} - \sigma_n t + \theta_n), \tag{3.1}$$

where $\mathbf{x}$ is the horizontal Cartesian co-ordinate, t the time, $\mathbf{k}_n$ a horizontal vector wave-number, σ_n the frequency, related to $\mathbf{k}_n$ by

$$\sigma_n^2 = g\,|\mathbf{k}_n| = gk_n$$

(where g is the acceleration of gravity). a_n and θ_n are amplitude and phase constants, chosen randomly so that $a_n\cos\theta_n$ and $a_n\sin\theta_n$ are jointly normal, with θ_n uniformly distributed and

$$\sum_{\mathbf{k}_n \ni d\mathbf{k}} \tfrac{1}{2}\overline{a_n^2} \doteqdot E(\mathbf{k})\,d\mathbf{k}.$$

Let $$a_n\cos\theta_n = \xi_n, \quad a_n\sin\theta_n = \xi_n'; \tag{3.2}$$

then we have $$\zeta^{(1)} = \sum_{n=1}^{N'} [\xi_n\cos(\mathbf{k}.\mathbf{x} - \sigma t) + \xi_n'\sin(\mathbf{k}.\mathbf{x} - \sigma t)],$$

which, if we write $\xi_{N'+n} = \xi_n'$, is of the form

$$\zeta^{(1)} = \sum_{i=1}^{2N'} \alpha_i\xi_i,$$

the α being constants for a fixed position and time. Also

$$\overline{\xi_i^2} = \tfrac{1}{2}\overline{a_i^2} \quad (i = 1, \ldots, 2N').$$

By the assumption of homogeneity we may consider the distribution of $\zeta^{(1)}$ at the special point $\mathbf{x} = 0$ and time $t = 0$; hence we may take

$$\alpha_i = \left\{\begin{matrix} 1 & (i = 1, \ldots, N'), \\ 0 & (i = N'+1, \ldots, 2N'). \end{matrix}\right\} \tag{3.3}$$

We can now make (3.3) correspond to the linear part of (2.1) by setting $N = 2N'$ and $\mathbf{u}_i = \mathbf{k}_i$, $\mathbf{u}_{N+i} = \mathbf{k}_i$ $(i = 1, \ldots, N')$. Moreover

$$\sum_{d\mathbf{u}} \overline{\xi_i^2} = \sum_{d\mathbf{u}} \tfrac{1}{2}\overline{a_i^2} = E(\mathbf{k})\, d\mathbf{u} \tag{3.4}$$

in each range $i = 1, \ldots, N'$, $i = N'+1, \ldots, 2N'$. So compared with (2.2) we have

$$F(\mathbf{u}) = E(\mathbf{k})$$

in each range also.

Corresponding to the free surface elevation $\zeta^{(1)}$ is a velocity potential

$$\phi^{(1)} = \sum_i b_i e^{k_i z} \sin \psi_i \qquad (b_i = a_1 \sigma_i / k_i).$$

However, $\zeta^{(1)}$ and $\phi^{(1)}$ are only first approximations. To satisfy the boundary conditions at the free surface to higher order one must add further terms in the series

$$\left.\begin{aligned} \zeta &= \zeta^{(1)} + \zeta^{(2)} + \ldots, \\ \phi &= \phi^{(1)} + \phi^{(2)} + \ldots, \end{aligned}\right\}$$

in which $\zeta^{(2)}$, $\phi^{(2)}$ contain terms proportional to the squares of the amplitudes; $\zeta^{(3)}$, $\phi^{(3)}$ contain terms proportional to the cubes of the amplitudes, and so on. The equations for $\phi^{(2)}$ and $\zeta^{(2)}$ are

$$\left.\begin{aligned} &\nabla^2 \phi^{(2)} = 0; \\ &\nabla \phi^{(2)} \to 0, \quad \text{when} \quad z \to -\infty; \\ &\left(\frac{\partial^2}{\partial t^2} + g\frac{\partial}{\partial z}\right)\phi^{(2)} = -\frac{\partial}{\partial t}(\nabla\phi^{(1)})^2 - \zeta^{(1)}\frac{\partial}{\partial z}\left(\frac{\partial^2}{\partial t^2} + g\frac{\partial}{\partial z}\right)\phi^{(1)} \quad \text{when} \quad z = 0, \end{aligned}\right\} \tag{3.5}$$

and

$$\zeta^{(2)} = -\frac{1}{g}\left[\frac{\partial \phi^{(2)}}{\partial t} + \tfrac{1}{2}(\nabla\phi^{(1)})^2 + \zeta^{(1)}\frac{\partial^2 \phi^{(1)}}{\partial z\, \partial t}\right]_{z=0} \tag{3.6}$$

It is assumed that the mean level $\overline{\zeta^{(2)}}$ is zero. Substituting for $\phi^{(1)}$ in the third of equations (3.5) we have

$$\begin{aligned} \left(\frac{\partial^2}{\partial t^2} + g\frac{\partial}{\partial z}\right)\phi^{(2)}_{z=0} = -\sum_{i,j} b_i b_j [&(\sigma_i - \sigma_j)(\mathbf{k}_i . \mathbf{k}_j + k_i k_j) \sin(\psi_i - \psi_j) \\ &+ (\sigma_i + \sigma_j)(\mathbf{k}_i . \mathbf{k}_j - k_i k_j) \sin(\psi_i + \psi_j)] \end{aligned}$$

and so

$$\begin{aligned} \phi^{(2)} = \sum_{i,j} b_i b_j \Bigg[&\frac{(\sigma_i - \sigma_j)(\mathbf{k}_i . \mathbf{k}_j + k_i k_j)}{(\sigma_i - \sigma_j)^2 - g\,|\mathbf{k}_i - \mathbf{k}_j|} e^{|\mathbf{k}_i - \mathbf{k}_j| z} \sin(\psi_i - \psi_j) \\ &+ \frac{(\sigma_i + \sigma_j)(\mathbf{k}_i . \mathbf{k}_j - k_i k_j)}{(\sigma_i + \sigma_j)^2 - g\,|\mathbf{k}_i + \mathbf{k}_j|} e^{|\mathbf{k}_i + \mathbf{k}_j| z} \sin(\psi_i + \psi_j)\Bigg]. \end{aligned}$$

Inserting this in (3.6) and substituting for b_i, b_j and σ_i, σ_j we find

$$\zeta^{(2)} = \sum_{i,j} \frac{a_i a_j}{(k_i k_j)^{\frac{1}{2}}} [\{B^-_{i,j} + B^+_{i,j} - \mathbf{k}_i . \mathbf{k}_j + (k_i + k_j)(k_i k_j)^{\frac{1}{2}}\} \cos \psi_i \cos \psi_j + (B^-_{i,j} - B^+_{i,j} - k_i k_j) \sin \psi_i \sin \psi_j], \quad (3.7)$$

where

$$B^-_{i,j} = \frac{(\sqrt{k_i} - \sqrt{k_j})^2 (\mathbf{k}_i . \mathbf{k}_j + k_i k_j)}{(\sqrt{k_i} - \sqrt{k_j})^2 - |\mathbf{k}_i . \mathbf{k}_j|}, \quad B^+_{i,j} = \frac{(\sqrt{k_i} + \sqrt{k_j})^2 (\mathbf{k}_i . \mathbf{k}_j - k_i k_j)}{(\sqrt{k_i} + \sqrt{k_j})^2 - |\mathbf{k}_i + \mathbf{k}_j|}. \quad (3.8)$$

Now at the point $\mathbf{x} = \mathbf{0}$, $t = 0$, the phase ψ_i reduces to θ_i and so

$$(a_i \cos \psi_i, a_i \sin \psi_i) = (\xi_i, \xi_{N+i}).$$

Hence to a second approximation we have

$$\zeta = \sum_i \alpha_i \xi_i + \sum_{i,j} \alpha_{ij} \xi_i \xi_j,$$

where α_{ij} is given by (3.3) and

$$\alpha_{ij} = \begin{cases} (k_i k_j)^{-\frac{1}{2}} \{B^-_{i,j} + B^+_{i,j} + \mathbf{k}_i . \mathbf{k}_j + (k_i + k_j)(k_i k_j)^{\frac{1}{2}}\} & \text{when} \quad i,j = 1, \ldots, N', \\ (k_i k_j)^{-\frac{1}{2}} (B^-_{i,j} - B^+_{i,j} - k_i k_j) & \text{when} \quad i,j = N'+1, \ldots, 2N', \\ 0 \quad \text{otherwise.} \end{cases} \quad (3.9)$$

The diagonal terms α_{ii} are given as the limit of the above expressions as $\mathbf{k}_j \to \mathbf{k}_i$. Then B^-_{ij} and B^+_{ij} both vanish and

$$\alpha_{ii} = \begin{cases} k_i & (i = 1, \ldots, N'), \\ -k_i & (i = N'+1, \ldots, 2N'). \end{cases}$$

Taking account of (3.3) we have then

$$\left. \begin{aligned} \kappa_1 &\doteqdot \alpha_{ii} V_i = \sum_{i=1}^{N'} k_i V_i - \sum_{i=1}^{N'} k_i V_i = 0, \\ \kappa_2 &\doteqdot \alpha_i \alpha_i V_i = \sum_{i=1}^{N'} V_i, \\ \kappa_3 &\doteqdot 6 \alpha_i \alpha_j \alpha_{ij} V_i V_j = 6 \sum_{i,j=1}^{N'} \alpha_{ij} V_i V_j. \end{aligned} \right\} \quad (3.10)$$

The first equation simply states that the mean surface level is zero, to second order, as was specified. The next two equations, in integral form, can be written

$$\kappa_2 \doteqdot \iint E(\mathbf{k})\, d\mathbf{k}, \quad \kappa_3 \doteqdot 6 \iiiint \mathrm{K}(\mathbf{k}, \mathbf{k}') E(\mathbf{k}) E(\mathbf{k}')\, d\mathbf{k} d\mathbf{k}', \quad (3.11)$$

where

$$\mathrm{K}(\mathbf{k}, \mathbf{k}') = \frac{1}{(kk')^{\frac{1}{2}}} \left[\frac{(\sqrt{k} - \sqrt{k'})^2 (\mathbf{k} . \mathbf{k}' + kk')}{(\sqrt{k} - \sqrt{k'})^2 - |\mathbf{k} - \mathbf{k}'|} + \frac{(\sqrt{k} + \sqrt{k'})^2 (\mathbf{k} . \mathbf{k}' - kk')}{(\sqrt{k} + \sqrt{k'})^2 - |\mathbf{k} + \mathbf{k}'|} - \mathbf{k} . \mathbf{k}' + (k + k')(kk')^{\frac{1}{2}} \right]. \quad (3.12)$$

If we take polar co-ordinates (k, θ) in the $\mathbf{k}$-plane and introduce the directional spectrum $F(\sigma, \theta)$ by

$$F(\sigma,\theta)\,d\sigma\,d\theta = E(\mathbf{k})\,d\mathbf{k} = E(\mathbf{k})\,k\,dk\,d\theta$$

so that

$$F(\sigma,\theta) = k\frac{dk}{d\sigma}E(\mathbf{k}) = \frac{k^2}{2\sigma}E(\mathbf{k}),$$

then we have

$$\kappa_2 = \iint F(\sigma,\theta)\,d\sigma\,d\theta, \quad \kappa_3 = 6\iiiint \mathrm{K}(\mathbf{k},\mathbf{k}')\,F(\sigma,\theta)\,F(\sigma',\theta')\,d\sigma\,d\sigma'\,d\theta\,d\theta'. \tag{3.13}$$

In the one-dimensional case when $F(\sigma, \theta)$ vanishes everywhere except when $\theta = 0$ the above expressions simplify very considerably. For when $\mathbf{k}'$ is parallel (or anti-parallel) to $\mathbf{k}$ we find

$$\mathrm{K}(\mathbf{k},\mathbf{k}') = \min(k, k')$$

and hence

$$\kappa_2 = \int F(\sigma)\,d\sigma, \quad \kappa_3 = 6\iint \min(k,k')\,F(\sigma)\,F(\sigma')\,d\sigma\,d\sigma', \tag{3.14}$$

where $F(\sigma) = \int F(\sigma,\theta)\,d\theta$ denotes the spectral density with regard to frequency. The above expression for κ_3 may be written

$$\kappa_3 = 12\iint_{k<k'} kF(\sigma)\,F(\sigma')\,d\sigma\,d\sigma' \tag{3.15}$$

$$= 12\int_0^\infty \left[\int_0^{\sigma'} kF(\sigma)\,d\sigma\right] F(\sigma')\,d\sigma', \tag{3.16}$$

where $k = \sigma^2/g$.

Let us examine the form of $\mathrm{K}(\mathbf{k}, \mathbf{k}')$ in the general case, when the angle between $\mathbf{k}$ and $\mathbf{k}'$ is equal to γ, say. Writing

$$(k+k')/2(kk')^{\frac{1}{2}} = \eta \geqslant 1$$

(by Schwarz's inequality) we find from (3.12) that

$$\mathrm{K}(\mathbf{k},\mathbf{k}') = (kk')^{\frac{1}{2}} f(\eta,\gamma), \tag{3.17}$$

where

$$f(\eta,\gamma) = \frac{(\eta-1)(1+c)}{(\eta-1)-(\eta^2-\frac{1}{2}-\frac{1}{2}c)^{\frac{1}{2}}} - \frac{(\eta+1)(1-c)}{(\eta+1)-(\eta^2-\frac{1}{2}+\frac{1}{2}c)^{\frac{1}{2}}} + (2\eta-c) \tag{3.18}$$

and $c = \cos\gamma$. It can be shown that $f(\eta, \gamma)$ is non-negative. For from (3.18)

$$f(\eta,\gamma) = \frac{2(\eta-1)(1+c)\,[(\eta-1)+(\eta^2-\frac{1}{2}-\frac{1}{2}c)^{\frac{1}{2}}]}{-(4\eta-c-3)} - \frac{2(\eta+1)(1-c)\,[(\eta+1)+(\eta^2-\frac{1}{2}+\frac{1}{2}c)^{\frac{1}{2}}]}{(4\eta-c+3)} + (2\eta-c).$$

Since all the factors in each expression are non-negative, the two radicals may be replaced by $(\eta^2)^{\frac{1}{2}} = \eta$ without diminishing the right-hand side. After some reduction we then find

$$f(\eta,\gamma) \geqslant \frac{(2\eta+c)(1-c^2)}{(4\eta-c)^2-9} \geqslant 0. \tag{3.19}$$

It follows that $\mathrm{K}(\mathbf{k}, \mathbf{k}')$ is non-negative, and that so also is κ_3, in the general case.

The form of $f(\eta, \gamma)$ can be seen from the curves in figure 1. The two extreme values are equal:

$$f(\eta, 0) = f(\eta, \pi) = \eta - (\eta^2 - 1)^{\frac{1}{2}}, \tag{3.20}$$

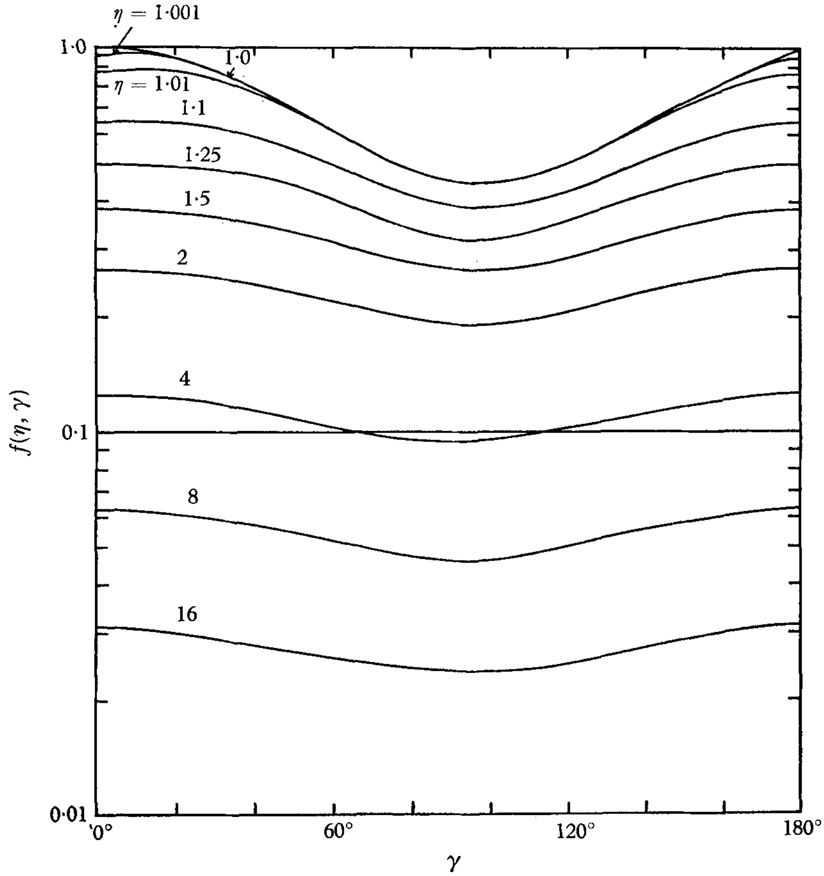

FIGURE 1. Graphs of $f(\eta, \gamma)$, defined by (3.17), for various values of η.

and for fixed values of η there is a minimum, at about $\eta = 90°$. However, the curves are not symmetrical about the mid-point of the range of η. For example, $f(1, \gamma)$ has a stationary point at $\gamma = 0$ but not at $\gamma = \pi$. Further, though it appears at first sight that $f(\eta, \gamma)$ never exceeds $f(\eta, 0)$, in fact $f(\eta, \gamma)$ is an increasing function of γ when γ is small and η is very close to 1. A numerical investigation shows that

$$0{\cdot}44 f(\eta, 0) \leqslant f(\eta, \gamma) \leqslant 1{\cdot}01 f(\eta, 0) \tag{3.21}$$

over all values of γ. From (3.13) and (3.17) it then follows that in general

$$0{\cdot}44 I \leqslant \kappa_3 \leqslant 0{\cdot}01 I, \tag{3.22}$$

where I denotes the integral on the right-hand side of (3.15) or (3.16). Further, writing

$$L = I / \kappa_2^{\frac{3}{2}}, \tag{3.23}$$

we deduce the following theoretical bounds for the skewness:

$$0{\cdot}44 L \leqslant \lambda_3 \leqslant 1{\cdot}01 L. \tag{3.24}$$

In the two-dimensional case we have always

$$\lambda_3 = L. \tag{3.25}$$

It will be seen from (2.28) that the next cumulant κ_4 generally involves the third- and fourth-order terms α_{ijk} and α_{ikjk}. These can be calculated in a similar way. In general, however, they will be weakly dependent on the time t, owing to resonant interactions between the wave components ξ_i (cf. Phillips 1960; Hasselmann 1960). Moreover α_{ijk}, for example, will generally be of the same order as $\min(k_i^2, k_j^2, k_k^2)$. Hence the convergence of the corresponding integrals will depend rather critically on the behaviour of the spectral density $E(\mathbf{k})$ at high wave-numbers. We shall not calculate the higher-order moments here, beyond remarking that according to the present analysis κ_4 and λ_4 are proportional to $\{E(\mathbf{k})\}^2$ and $\{E(k)\}$, respectively. Hence if the integrals substantially converged over the region in which viscous damping was negligible then λ_4 would be of the same order as λ_3^2.

4. Comparison with observation

An extensive study of the power spectra of water waves over short fetches, and of the corresponding statistical distribution of surface elevation, has been made by Kinsman (1960). In the second column of table 1 are shown Kinsman's observed values of κ_2, and in the third column the value calculated from the power spectra* using equation (3.11). The first group of calculations, from records 009 to 067, are based on estimates of spectral density at frequencies of 0(0·1) 2·5 c/s; the second group are based on estimates at more closely spaced intervals from 0(0·05) 2·5 c/s (not available for records 009 to 028). It will be seen that the agreement between the observed and calculated values of κ_2 is within 5 % except in the case of two records, 028 and 087 (for which there seems no obvious explanation). There is only a slight difference, of the order of 1 %, in record 067, between the values of calculated from the less closely spaced, and from the more closely spaced, spectral estimates.

In the fourth column of table 1 are listed the values of the skewness coefficient λ_3, as observed by Kinsman.† It will be seen at once that all except three of the observations are positive, as predicted, and of the three negative values, two are very small.

In the next column of table 1 are shown the values of L calculated from equation (3.20), that is to say the theoretical values of λ_3 if the spectrum were uni-directional. (The two theoretical estimates for record 067, which differ by about 12 %, suggest that the estimates derived from the more closely spaced spectral estimates are significantly more accurate.) The ratio λ_3/L is shown in the sixth column of table 1. According to equation (3.24) this ratio should lie between 0·44 and 1·01. Out of the total of 24 records it will be seen that 18 satisfy the inequality $\lambda_3/L \leqslant 1{\cdot}01$ and that 15 satisfy $\lambda_3/L \geqslant 0{\cdot}44$.

In the last column of table 1 are shown the observed values of λ_4 as found by

* Kinsman tabulates $P(f)$ where f = frequency in c/s. He uses a slightly different definition of the power spectrum, so that $P df = \frac{1}{2}F(\sigma)\,d\sigma$.

† In table 5.10 of Kinsman (1960) it is $\frac{1}{2}\lambda_3$ that is tabulated.

Kinsman (who tabulates $\frac{1}{2}\lambda_4$). Although some of these values are of order λ_3^2, as might be expected, there are several values of nearly 0·4, which is unexpectedly high. Part of the variability in λ_4 may no doubt be attributed to the finite size of the sample; but probably it reflects also the sensitivity of the integrals to the high-frequency end of the spectrum, which is not included in the measured values.

Record	κ_2 (cm^2) obs.	κ_2 (cm^2) th.	λ_3 obs.	L	λ_3/L	λ_4 obs.
009	8·45	8·39	0·344	0·284	1·21	0·092
010	8·94	8·94	0·286	0·293	0·98	−0·030
011	10·82	10·77	0·192	0·234	0·82	−0·250
012	8·45	8·46	0·364	0·273	1·33	0·202
017	3·30	3·24	0·350	0·274	1·28	0·100
018	4·12	4·09	0·438	0·257	1·70	0·366
027	3·87	3·83	0·316	0·264	1·20	−0·392
028	3·13	3·77	0·356	0·204	1·75	0·118
067	4·99	4·95	0·164	0·282	0·58	−0·014
067	4·99	5·02	0·164	0·258	0·64	−0·014
068	5·57	5·60	0·174	0·260	0·67	0·050
069	7·71	7·71	0·138	0·248	0·56	0·414
070	7·23	7·23	0·054	0·243	0·22	0·090
075	9·65	9·62	0·202	0·249	0·81	0·086
076	7·37	7·37	0·184	0·240	0·77	−0·062
081	3·46	3·40	0·058	0·169	0·34	−0·130
082	3·64	3·59	0·068	0·196	0·35	−0·232
083	7·57	7·37	−0·004	0·192	−0·02	−0·202
084	6·64	6·59	0·088	0·217	0·41	0·048
085	7·91	7·77	−0·010	0·203	−0·05	−0·448
086	7·72	7·71	0·022	0·223	0·10	−0·156
087	3·45	7·45	0·010	0·068	0·15	0·330
088	4·06	4·05	−0·092	0·177	−0·52	0·300
093	9·34	9·33	0·288	0·336	0·86	0·432
094	11·30	11·29	0·272	0·363	0·75	0·046

TABLE 1. Comparison of observed and theoretical coefficients of distributions of surface elevation.

The observed distributions $p(\zeta)$ themselves have been compared by Kinsman (1960) with the expressions

$$[\exp(-\tfrac{1}{2}\zeta^2/\kappa_2)]\,(2\pi\kappa_2)^{-\frac{1}{2}}\,[1+\tfrac{1}{6}\lambda_3 H_3+\tfrac{1}{24}\lambda_4 H_4] \tag{4.1}$$

(see figures A III 2.01–2.24 of Kinsman 1960) and it is found that the observations are an appreciably better fit to (4.1) than to the corresponding Gaussian distributions, from which (4.1) differs by terms of order λ_3. Equation (4.1) differs from the theoretical distribution (2.20) by the term

$$[\exp(-\tfrac{1}{2}\zeta^2/\kappa_2)]\,(2\pi\kappa_2)^{-\frac{1}{2}}\,\tfrac{1}{72}\lambda_3^2 H_6 \tag{4.2}$$

which is of order λ_3^2. Since the maximum value of

$$|(2\pi)^{-\frac{1}{2}}\exp(-\tfrac{1}{2}\zeta^2/\kappa_2)\,\tfrac{1}{72}H_6|$$

is equal to 0·083 at $\zeta = 0$, and since the maximum value of λ_3, from table 1, is 0·438 ($\lambda_3^2 = 0{\cdot}192$) it will be seen that the terms (4·2) are in fact rather small. It is found that they make no appreciable difference to the theoretical distributions (for the values of λ_3 observed) and that the agreement with observation is not significantly improved. Equation (2.20) is indeed a significant improvement over the Gaussian distribution, but this is brought about mainly by the term in λ_3 which is already included in (4.1).

5. Joint distribution of two non-linear variables

The joint distribution of two or more variables of type similar to (2.1) may be investigated in an exactly similar way. In the present section we shall evaluate the distribution for two such variables. This will enable us, in the following section, to evaluate the joint distribution of the two components of surface slope in a random sea.

Consider two variables ζ, η defined by

$$\left.\begin{aligned}\zeta &= \alpha_i\xi_i + \alpha_{ij}\xi_i\xi_j + \alpha_{ijk}\xi_i\xi_j\xi_k + \ldots,\\ \eta &= \beta_i\xi_i + \beta_{ij}\xi_i\xi_j + \beta_{ijk}\xi_i\xi_j\xi_k + \ldots,\end{aligned}\right\} \tag{5.1}$$

where α_i, α_{ij}, ... and β_i, β_{ij}, ... are constants and the ξ_i are defined as before. We denote by A_p and B_q the terms $\alpha_{ij\ldots l}$ and $\beta_{ij\ldots m}$ which contain respectively p and q suffices, and by $(A_pB_q\ldots)$ the irreducible part of the mean product

$$\overline{(A_p\xi_{i_1}\xi_{j_1}\ldots\xi_{l_1})(B_q\xi_{i_2}\xi_{j_2}\ldots\xi_{m_2})\ldots}.$$

For example

$$\left.\begin{aligned}(A_1B_1) &= \alpha_i\beta_iV_i, \quad (A_1^nB_1^m) = 0 \quad (n+m > 2);\\ (A_1B_1B_2) &= 2\alpha_i\beta_j\beta_{ij}V_iV_j, \quad (A_1B_3) = 3\alpha_i\beta_{ijj}V_iV_j,\\ (A_1^2B_2) &= 2\alpha_i\alpha_j\beta_{ij}V_iV_j.\end{aligned}\right\} \tag{5.2}$$

The the joint moments of ζ and η may be written down by inspection. Thus

$$\mu_{11} = \overline{\zeta\eta} = \sum_{p,q}[(A_pB_q) + (A_p)(B_q)],$$

$$\mu_{21} = \overline{\zeta^2\eta} = \sum_{p,q}[(A_pA_qB_r) + (A_pA_q)(B_r) + 2(A_p)(A_qB_r) + (A_p)(A_q)(B_r)],$$

and in general

$$\mu_{n\,m} = \overline{\zeta^n\eta^m} = \sum_{p,q,\ldots,s} C(\varpi)\,\varpi(i_1, j_1;\, i_2, j_2;\, \ldots), \tag{5.3}$$

where ϖ denotes a grouping of $A_p, \ldots, B_q, \ldots$ into unordered sets containing i_1 of A_p and j_1 of B_q; i_2 of A_p and j_2 of B_q; etc., with $(i_1+i_2+\ldots) = n$ and $(j_1+j_2+\ldots) = m$; and ϖ is the number of distinct ways of choosing such a grouping.

If $p(\zeta, \eta)$ denotes the joint density of ζ and η the moment-generating function for the joint distribution is defined by

$$\phi(it, is) = \iint p(\zeta, \eta)\exp(it\zeta + is\eta)\,d\zeta\,d\eta = \sum_{i,j}\frac{\mu_{ij}}{i!\,j!}(it)^i(is)^j$$

and the cumulant-generating function is defined by

$$\mathrm{K}(it, is) = \log \phi(it, is) = \sum_{(i,j)\neq(0,0)} \frac{\kappa_{ij}}{i!\,j!}(it)^i (is)^j,$$

where in the summation i and j take all pairs of non-negative integral values except $(i, j) = (0, 0)$. Thus we have

$$\phi(it, is) = \exp[\mathrm{K}(it, is)] = \sum_{r=0}^{\infty} \frac{1}{r!}\left[\sum_{(i,j)\neq(0,0)} \frac{\kappa_{ij}}{i!\,j!}(it)^i (is)^j\right]^r$$

and by equating coefficients of $t^m s^n$ in this expression we have

$$\mu_{mn} = \Sigma C(i_1, j_1;\ i_2, j_2;\ \ldots)\, \kappa_{i_1,j_1} \kappa_{i_2,j_2} \ldots,$$

where $C(i_1, j_1;\ i_2, j_2;\ \ldots)$ is the same constant as in (5.3). Hence we have simply

$$\kappa_{ij} = \sum_{p_1\ldots p_i q_1\ldots q_j} (A_{p_1} \ldots A_{p_i} B_{q_1} \ldots B_{q_j}). \tag{5.4}$$

In particular, $\kappa_{i0} = \kappa_i$, which is given as far as the terms in V^3 by equation (2.13). Similarly

$$\left.\begin{aligned}
\kappa_{11} &= (A_1B_1) + [(A_1B_3) + (A_2B_2) + (A_3B_1)] \\
&\quad + [(A_1B_5) + (A_2B_4) + \ldots + (A_5B_1)], \\
\kappa_{21} &= [(A_1^2B_2) + 2(A_1A_2B_1)] + [(A_1^2B_4) + 2(A_1A_2B_3) \\
&\quad + 2(A_1A_3B_2) + (A_2^2B_2) + 2(A_1A_4B_1) + 2(A_2A_3B_1)], \\
\kappa_{22} &= [2(A_1^2B_1B_3) + (A_1^2B_2^2) + 4(A_1A_2B_1B_2) + (A_2^2B_1^2) + 2(A_1A_3B_1^2)], \\
\kappa_{31} &= [(A_1^3B_3) + 3(A_1^2A_2B_2) + 3(A_1A_2^2B_1)],
\end{aligned}\right\} \tag{5.5}$$

etc. It is evident that in general κ_{ij} is of order $V^{(i+j-1)}$.

The joint density $p(\zeta, \eta)$ can now be found from

$$\begin{aligned}
p(\zeta,\eta) &= \frac{1}{(2\pi)^2}\iint \phi(it, it') \exp[-i(\zeta t + \eta t')]\, dt\, dt' \\
&= \frac{1}{(2\pi)^2}\iint \exp\left[-i\zeta t - i\eta t' + \sum_{(i,j)\neq(0,0)} \frac{\kappa_{ij}}{i!\,j!} t^i t'^j\right] dt\, dt' \\
&= \frac{1}{(2\pi)^2}\iint \exp[\{(\kappa_{10} - \zeta)\, it + (\kappa_{01} - \eta)\, it'\} - \tfrac{1}{2}\{\kappa_{20}t^2 + 2\kappa_{11}tt' + \kappa_{02}t'^2\} \\
&\quad + \tfrac{1}{6}\{\kappa_{30}(it)^3 + 3\kappa_{21}(it)^2(it') + 3\kappa_{12}(it)(it')^2 + \kappa_{03}(it')^3\} + \ldots]\, dt\, dt'.
\end{aligned} \tag{5.6}$$

Writing

$$t = u/\kappa_{20}^{\frac{1}{2}}, \quad (\zeta - \kappa_{10}) = f\kappa_{20}^{\frac{1}{2}},$$

$$t' = u'/\kappa_{02}^{\frac{1}{2}}, \quad (\eta - \kappa_{01}) = f'\kappa_{02}^{\frac{1}{2}},$$

and

$$\lambda_{ij} = \kappa_{ij}/(\kappa_{20}^{i}\kappa_{02}^{j})^{\frac{1}{2}},$$

we have

$$\begin{aligned}
p(\zeta,\eta) = \frac{1}{(2\pi)^2(\kappa_{20}\kappa_{02})^{\frac{1}{2}}} &\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \exp[-i(fu + f'u') - \tfrac{1}{2}(u^2 + 2\lambda_{11}uu' + u'^2)] \\
&\times \exp[\tfrac{1}{6}i^3\{\lambda_{30}u^3 + 3\lambda_{21}u^2u' + \ldots\} + \ldots]\, du\, du'.
\end{aligned} \tag{5.7}$$

Now

$$\frac{1}{2\pi}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\left[-i(fu+f'u')-\tfrac{1}{2}(u^2+2\rho uu'+u'^2)\right](iu)^m(iu')^n\,du\,du'$$

$$=\frac{(-1)^{m+n}}{2\pi}\frac{\partial^m}{\partial f^m}\frac{\partial^n}{\partial f'^n}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\exp\left[-i(fu+f'u')-\tfrac{1}{2}(u^2+2\rho uu'+u'^2)\right]du\,du'$$

$$=\frac{(-1)^{m+n}}{(1-\rho^2)^{\frac{1}{2}}}\frac{\partial^m}{\partial f^m}\frac{\partial^n}{\partial f'^n}\exp\left[-\tfrac{1}{2}(f^2+2\rho ff'+f'^2)/(1-\rho^2)\right]$$

$$=(1-\rho^2)^{-\frac{1}{2}}H_{mn}(f,f';\rho)\exp\left[-\tfrac{1}{2}(f^2-2\rho ff'+f'^2)/(1-\rho^2)\right]$$

say, where H_{mn} is a two-dimensional analogue of the Hermite polynomial. Thus

$$\left.\begin{aligned}H_{00}&=1,\\ H_{10}&=(f-\rho f')/(1-\rho^2)^{\frac{1}{2}},\\ H_{01}&=(f'-\rho f)/(1-\rho^2)^{\frac{1}{2}},\\ H_{20}&=(f-\rho f')^2/(1-\rho^2)-1,\\ H_{11}&=(f-\rho f')(f'-\rho f)/(1-\rho^2)+\rho,\\ H_{02}&=(f'-\rho f)^2/(1-\rho^2)-1,\end{aligned}\right\}\tag{5.8}$$

etc., and when $\rho = 0$

$$H_{nm}(f,f';0)\equiv H_n(f)\,H_m(f').$$

So writing $\lambda_{11} = \rho$ in (5.7) we have

$$p(\zeta,\eta)=\{2\pi(\kappa_{20}\kappa_{02}-\kappa_n)^{\frac{1}{2}}\}^{-1}\exp\left[-\tfrac{1}{2}(f^2-2\rho ff'+f'^2)/(1-\rho^2)\right]$$
$$\times\left[1+\tfrac{1}{6}(\lambda_{30}H_{30}+3\lambda_{21}H_{21}+\ldots)+\ldots\right].\tag{5.9}$$

In the first approximation, when terms of order $V^{\frac{1}{2}}$ are neglected,

$$p(\zeta,\eta)=\{2\pi(\kappa_{20}\kappa_{02}-\kappa_{11}^2)^{\frac{1}{2}}\}^{-1}\exp\left[-\tfrac{1}{2}(f^2-2\rho ff'+f'^2)/(1-\rho^2)\right],$$

where $f = \zeta/\kappa_{20}^{\frac{1}{2}}$, $f' = \eta/\kappa_{02}^{\frac{1}{2}}$ and

$$\kappa_{20}=\alpha_i\alpha_i,\quad \kappa_{02}=\beta_i\beta_i,\quad \kappa_{11}=\alpha_i\beta_i.$$

This is the familiar Gaussian bivariate distribution.

In the second approximation, taking into account $V^{\frac{1}{2}}$ but neglecting V, the mean of the distribution is shifted to

$$(\bar{\zeta},\bar{\eta})=(\kappa_{10},\kappa_{01})$$

and the abscissa is multipied by the factor

$$\left[1+\tfrac{1}{6}(\lambda_{30}H_{30}+3\lambda_{21}H_{21}+\ldots)\right]$$

introducing various kinds of skewness, specified by the parameters λ_{30}, λ_{21}, λ_{12} and λ_{03}.

6. Application to the distribution of surface slopes

We shall now apply the results of the last section to evaluate the joint distribution of the two components of slope of a random sea surface.

Suppose that the surface elevation is given by equation (2.1). Then on partial differentiation with respect to x and y respectively we have

$$\left.\begin{aligned}\partial\zeta/\partial x&=\beta_i\xi_i+\beta_{ij}\xi_i\xi_j+\ldots,\\ \partial\zeta/\partial y&=\gamma_i\xi_i+\gamma_{ij}\xi_i\xi_j+\ldots,\end{aligned}\right\}\tag{6.1}$$

where, if (u_i, v_i) denote the components of the wave-number $\mathbf{k}_i$, we have

$$\beta_i = \alpha_i u_i, \quad \gamma_i = \alpha_i v_i.$$

Using the form of α_i as given by equation (3.3) we see that β_i and γ_i, when expressed as vectors, have the form

$$\left.\begin{aligned}(\beta_i) &= (0, 0, \ldots, 0;\ -u_1, -u_2, \ldots, -u_N),\\ (\gamma_i) &= (0, 0, \ldots, 0;\ -v_1, -v_2, \ldots, -v_N).\end{aligned}\right\} \tag{6.2}$$

Similarly from (3.9) we see that β_{ij}, when expressed as a matrix, has the form

$$(\beta_{ij}) = \begin{pmatrix} \mathbf{0} & \mathbf{M} \\ \mathbf{M}' & \mathbf{0} \end{pmatrix}, \tag{6.3}$$

where

$$\mathbf{M} = \begin{pmatrix} (u_1\alpha_{N+1,N+1} - u_1\alpha_{11}) & \cdots & (u_1\alpha_{N+1,2N} - u_N\alpha_{1,N}) \\ \vdots & & \vdots \\ (u_N\alpha_{2N,N+1} - u_1\alpha_{N,1}) & \cdots & (u_N\alpha_{2N,2N} - u_N\alpha_{N,N}) \end{pmatrix}; \tag{6.4}$$

(γ_{ij}) has an exactly similar form, except that v_i replaces u_i.

Let C_r denote the term $\gamma_{ij\ldots n}$ which contains just r suffices. We see now that the results of § 6 are applicable to the two non-linear variables $\partial\zeta/\partial x$, $\partial\zeta/\partial y$ provided that we replace A_p, B_q by B_p, C_q, respectively. In particular we have for the first few cumulants of the joint distribution (retaining only the lowest-order terms) the following: for the second-order cumulants

$$\left.\begin{aligned}\kappa_{20} &= (B_1^2) = u_i^2 V_i = \iint u^2 E(\mathbf{k})\, d\mathbf{k},\\ \kappa_{11} &= (B_1 C_1) = u_i v_i V_i = \iint uv E(\mathbf{k})\, d\mathbf{k},\\ \kappa_{02} &= (C_1^2) = v_i^2 V_i = \iint v^2 E(\mathbf{k})\, d\mathbf{k}.\end{aligned}\right\} \tag{6.5}$$

Each of these is proportional to $E(\mathbf{k})$ (or V). In the expressions for the *third-order* cumulants κ_{30}, κ_{21}, κ_{12}, κ_{03} it will be seen from (6.2) and (6.3) that all the leading terms vanish identically,

$$(B_1^2 B_2) = (B_1^2 C_2) = (C_1^2 B_2) = (C_1^2 C_2) = 0.$$

Hence in the joint distribution of the surface slopes, *the third-order cumulants are of order V^3 at least*. The terms of next lowest order are

$$\left.\begin{aligned}\kappa_{30} &= 3(B_1^2 B_4) + 6(B_1 B_2 B_3),\\ \kappa_{21} &= (B_1^2 C_4) + 2(B_1 B_2 C_3) + 2(B_1 B_3 C_2) + 2(B_1 B_4 C_1) + 2(B_2 B_3 C_1),\end{aligned}\right\} \tag{6.6}$$

with similar expressions for κ_{12} and κ_{03}.

If the distribution of slopes were symmetrical about the mean, then clearly all cumulants of odd order, such as the third-order cumulants, would necessarily vanish identically. This would certainly follow if, for example, it were possible to reverse the direction of time (so that forward slopes became rear slopes, and vice versa) without altering the statistics of the surface. However, we know in fact that the time cannot be reversed, even for free, undamped waves, for it has been shown that there exists a slow transfer of energy from one part of the spectrum to another (Phillips 1960; Hasselmann 1960). This transfer is represented by certain terms of the third-order which occur in A_3 and hence in B_3 and C_3.* Since either B_3 or C_3 occur in (6.6) we expect that the third-order cumulants are indeed of order V^3.

Similarly from (5.5) we have for the leading terms in the cumulants of fourth order

$$\left.\begin{aligned}\kappa_{40} &= 4(B_1^3B_3)+6(B_1^2B_2^2),\\ \kappa_{31} &= (B_1^3C_3)+3(B_1B_2^2C_1),\\ \kappa_{22} &= (B_1^2C_2^2)+2(B_1^2C_2C_3)+4(B_1B_2C_1C_2)+2(B_1B_3C_1^2)+(B_2^2C_1^2),\end{aligned}\right\} \quad (6\cdot7)$$

etc. These also are of order V^3.

It follows that the coefficients of skewness

$$\lambda_{30} = \kappa_{30}/(\kappa_{20})^{\frac{3}{2}}, \quad \lambda_{21} = \kappa_{21}/\kappa_{20}\kappa_{02}^{\frac{1}{2}}, \quad \text{etc.},$$

are each of order $V^{\frac{3}{2}}$ in general, and that the coefficients of kurtosis

$$\lambda_{40} = \kappa_{40}/\kappa_{20}^2, \quad \lambda_{31} = \kappa_{31}/\kappa_{20}^{\frac{3}{2}}\kappa_{02}^{\frac{1}{2}}, \quad \text{etc.},$$

are each of order V.

It should be emphasized that the present model of the sea surface is a model of 'free' waves, in which not only is the viscous damping neglected but it is also assumed that the stresses at the free surface are identically zero. On the other hand, both the viscosity, and also the stresses due to the action of the atmosphere on the surface, may be expected to produce some asymmetry in the wave profile. Since for free waves the skewness is theoretically of such a high order, $\sim (\kappa_{20}+\kappa_{02})^{\frac{3}{2}}$, the actual skewness of sea waves may be a rather sensitive indicator of energy transfer to the water, or dissipation of energy in the medium.

7. Comparison with observation

It follows that in any comparison of the theoretical results with observation, some consideration must be given to whether the conditions of the theory (i.e. free, undamped surface waves) are actually satisfied. If there is any appreciable transfer of energy from the atmosphere to the sea surface, or if there is a considerable contribution to the slope distribution from the very short waves, which are the most highly damped, then the free-wave model cannot be expected to apply.

In model experiments, Cox (1958) has shown that in winds of between 3 and 12 m/sec a large part of the contribution to the mean square slope (observed optically) is associated with frequencies above 10 c/s, and hence with waves that are influenced predominantly by surface tension.† If the analysis of the preceding section were modified so as to include the effect of surface tension, it would still be found that the third-order cumulants were of fourth order in the wave amplitudes. On the other hand it is unlikely that, even with surface tension

* If the spectrum is one-dimensional, these terms vanish.

† The frequency of waves having the minimum phase-velocity is about 11 c/s.

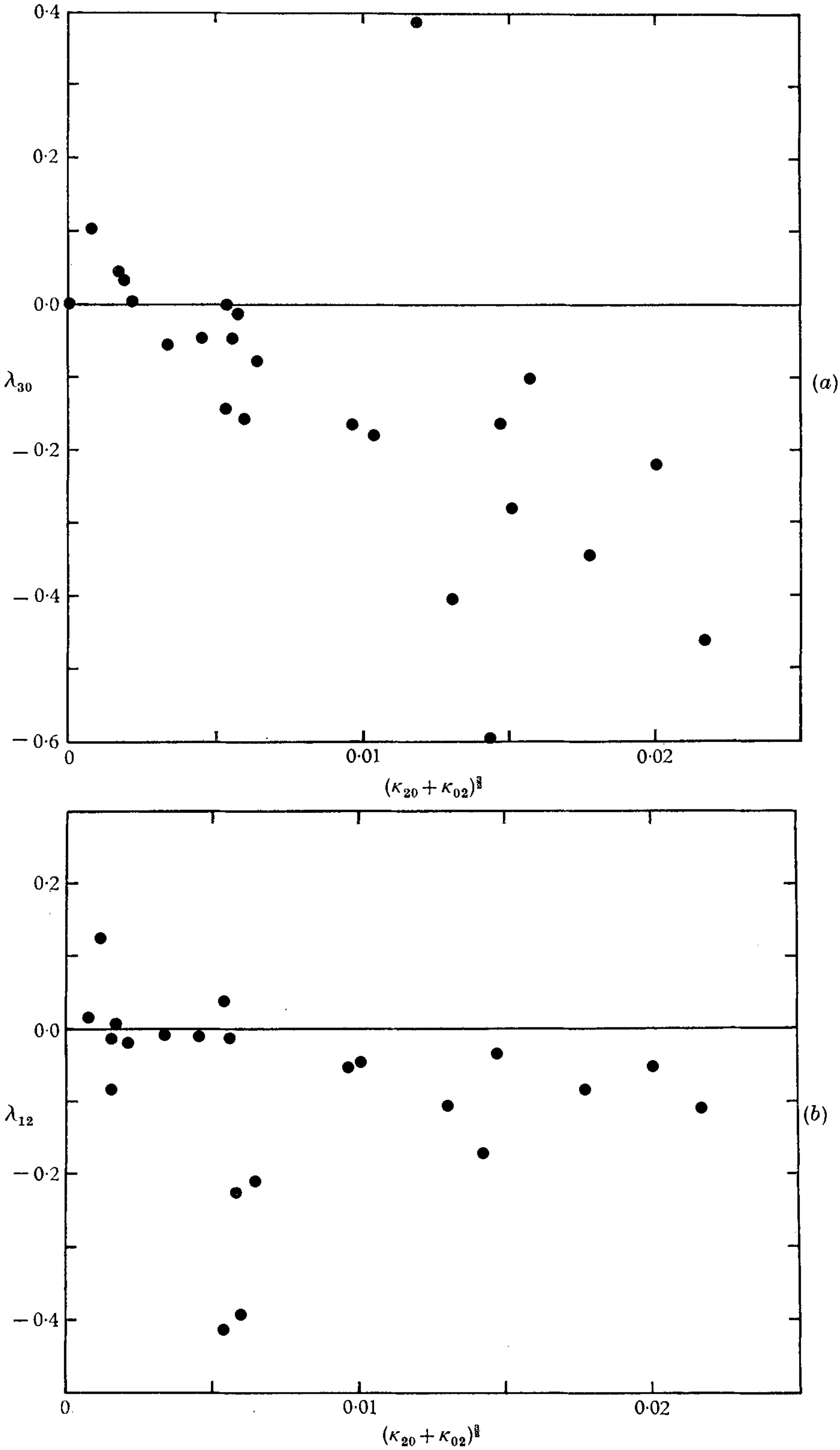

FIGURE 2. Observed values of the coefficients of skewness, plotted against (mean-square slope)$^{\frac{3}{2}}$, from the data of Cox & Munk (1956).

included, the perturbation analysis of § 5 would apply in such a situation, for the following reasons.

First, as the wavelength decreases, the short waves become increasingly influenced by viscosity, the time-constant being equal to $(2\nu k^2)^{-1}$ or about $0{\cdot}71\lambda^2$ where λ is the wavelength in cm (see Lamb 1932, § 348). In § 5 this damping was entirely neglected.

Secondly, it is unlikely that a linear first approximation is appropriate when the wavelengths and time-constants of the short waves are small compared with the orbital displacements and the periods of the longer waves. It is known also that there is a direct transfer of energy to capillary waves from the steep crests of the gravity waves (Longuet-Higgins 1963), which can hardly be described in terms of the linear perturbation scheme.

Thirdly, the direct action of the airflow over the water surface, which is perhaps most important at high frequencies, has been neglected.

Hence the conclusions of § 5 cannot be expected to apply to the observed slope distribution, certainly if the wind exceeds 3·18 m/sec and probably if it exceeds the minimum phase-velocity of 19 cm/sec. An exception may occur in the presence of oil slicks, which are known to remove the energy in the highest wave-numbers.

In the measurements of surface slope made by Cox & Munk (1956), the local wind-speed ranged from 72 to 1380 cm/sec. Nevertheless, using the data given in table 1 of Cox & Munk's paper we have calculated the coefficients of skewness in terms of the quantities defined in their paper, that is (if the x-axis is taken in the direction of the wind):

$$\left.\begin{aligned} \lambda_{30} &= c_{03} = -6\sigma_u^3(a_1'+a_3), \\ \lambda_{12} &= c_{21} = -2\sigma_u\sigma_c^2(a_1'-3a_3). \end{aligned}\right\} \tag{7.1}$$

Also
$$\kappa_{20} = \sigma_u^2, \quad \kappa_{02} = \sigma_c^2. \tag{7.2}$$

A plot of λ_{30} and λ_{12} against $(\kappa_{20}+\kappa_{02})^{\frac{3}{2}}$ is shown in figure 2(*a*) and (*b*), for those cases when the sea surface was free of slicks.

From figure 2(*a*) it does appear that λ_{30} is approximately proportional to $(\kappa_{20}+\kappa_{02})^{\frac{3}{2}}$, as predicted. However, since the order of magnitude of λ_{30} is about twenty times that of $(\kappa_{20}+\kappa_{02})^{\frac{3}{2}}$ it appears more likely that the theory does not really apply to these observations.

On the other hand, Cox & Munk observed that in the presence of oil slicks the coefficients of skewness were very greatly reduced, and were in fact so small as not to be measurable by their technique. Though not conclusive, this observation is certainly consistent with the theoretical result.

8. Conclusions

We have rederived the statistical distribution of weakly non-linear variables of the type given by equation (2.1) or equations (5.1). The distribution of such quantities is shown to be Gaussian in the first approximation, and in successively higher approximations to be given by Edgeworth's form of a Gram–Charlier series, as in equations (2.20) and (5.9), respectively. These series differ from the series used, for example, by Kinsman and by Cox & Munk as empirical fits to observed data, but the differences occur only in the third and higher approximations; they are practically negligible in the cases considered.

The coefficients in the distribution depend essentially on the cumulants of the original variables, which can be calculated simply in terms of the constants in (2.1) or (5.1). If one assumes that the sea surface consists of free, undamped waves then it can be shown that the skewness of the surface elevation is always positive and lies between the two bounds (3.24). This agrees, for the most part, with Kinsman's observations. The skewness increases proportionally to the R.M.S. surface slope s. On the other hand the skewness of the *slope* distribution is of a higher order and increases proportionally to s^3. While this prediction is consistent with the observations of Cox & Munk, nevertheless, if the local wind is appreciable the skewness may be largely affected by energy transfer from air to water, and by viscous dissipation.

I am indebted to Dr K. Hasselmann and Dr C. S. Cox for valuable comments on a first draft of this paper, and to Mr C. L. Gulliver and Mrs W. Wilson for assistance with the calculations for table 1 and the diagrams.

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CORRIGENDUM

'On the sensitivity of heat transfer in the stagnation-point boundary layer to free-stream vorticity', by S. P. Sutera, P. F. Maeder and J. Kestin, *J. Fluid Mech.* **16**, 1963, pp. 497–520.

(i) In the summary on page 497, replace 'disturbed' by 'distributed'.

(ii) Page 498, line 15. Insert the word 'oscillating' after 'harmonically'.

(iii) Page 501, equation (6*a*). Insert x to make right-hand side read as

$$\to ax.$$

(iv) Page 508, figure 1. The inner ordinate scale should be given the additional label $\phi''_{(1)}$, and the lowest of the three curves should be labelled $\phi''_{(1)}$ instead of $\phi'_{(1)}$.

(v) Page 512, 11 lines up. Reverse the inequality signs so that it reads as

$$2{\cdot}4 > \eta > 0{\cdot}9.$$

(vi) Page 516, figure 8. Replace 'v_0' by 'V_0'.